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WAVE COHERENCE, COUPLING POWER AND STATISTICAL ENERGY ANALYSIS

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The effects of the coherence of waves incident upon the two sides of a coupling between two subsystems are considered, with reference to the evaluation of the coupling power between the subsystems. These effects are due to waves travelling out from the coupling being reflected from other parts of the structure, these reflections being later incident upon the coupling. It is seen that when frequency or ensemble averaged, the net coherence effects can be very substantial. However, these effects are ignored in the normal wave description of the energy flow through a coupling, such as that used in statistical energy analysis (SEA), and are a major source of error in wave-based approaches to SEA. For two, one-dimensional subsystems, a parameter γ is identified which relates wave transmission and dissipation effects and quantifies the strength of coupling between the two subsystems. When the coupling is strong ($\gamma > 1$), transmission effects dominate and the net effects of coherence are large. When the coupling is weak ($\gamma < 1$), dissipation effects dominate, coherence effects are negligible on average and normal SEA approaches give accurate estimates of the coupling power. More general cases of coupled one- and two-dimensional subsystems are then considered. The effects of coherence and the use of this parameter as a measure of strength of coupling are discussed, with coherence effects being reduced in geometrically irregular systems.

1. INTRODUCTION

The vibration of complex structures at high frequencies is often described by the flow of vibrational energy through the structure, this structure-borne energy being transported by waves. The system may be divided into a number of subsystems coupled together, the coupling power at any such coupling being inferred from the powers in the incident wave trains and the transmission properties of the coupling. This paper concerns certain aspects of this coupling power, with particular reference to statistical energy analysis (SEA) [1–3]. In SEA the response of a system is described in terms of the time, space and (usually) frequency average subsystem energies, input powers and coupling powers. This is in contrast to detailed, deterministic analysis, such as by way of a finite element model, in which the response at specific points at specified frequencies is calculated. This is partly due to the computational effort required at high frequencies and, more profoundly, to a recognition of the fact that uncertainty exists in one's knowledge of the exact physical and geometric properties, the boundary conditions and even the equation of motion of the structure. At high frequencies the behaviour is sensitive to this uncertainty and therefore exact, deterministic predictions are of dubious value.

In a wave approach, the coupling power between two subsystems arises from the transmission of energy associated with the two incident wave trains. Normally these wave

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trains are considered to be incoherent, at least when frequency and/or ensemble averaged, so that the net effects of coherence are negligible. This is particularly true of the traditional estimate of coupling power in SEA. This paper considers these coherence effects in more detail. It is seen that, for two coupled one-dimensional subsystems, when the coupling is weak, in a certain sense, coherence effects are negligible, while they are far from negligible for strong coupling—in this latter case the SEA estimate of coupling power is inaccurate. The strength of coupling between the subsystems is quantified by a parameter γ which relates transmission and damping effects. Certain other cases are then discussed and measures of coupling strength proposed. The discussion concentrates on systems comprising two subsystems, but some remarks concerning the effects of a third subsystem are made—these generally reduce the effects of coherence. For two- and three-dimensional subsystems similar results are seen to hold if the systems are regular. Irregularity in the system scatters an incident wave into different directions and generally reduces coherence effects.

To illustrate these remarks, consider the line of coupling between two subsystems a and b as shown in Figure 1. Waves are incident on both sides of the coupling at an angle θ. (It is assumed in this paper that the subsystems have the same wavenumber. If this is not the case, then one should consider incident angles θ_a and θ_b which are related by Snell's law, such that the incident waves have the same trace wavenumber.) The coupling is described by reflection and transmission coefficients $r(\theta)$ and $t(\theta)$ such that, at a frequency ω, the amplitudes of the incident and scattered waves are related by

$$a^- = r_{aa}a^+ + tb^-, \qquad b^+ = ta^+ + r_{bb}b^-, \tag{1}$$

where $a^\pm$ and $b^\pm$ are the wave amplitudes, the $\pm$ superscript indicating the direction of propagation. The coupling is assumed to be conservative. Furthermore, wave amplitudes will be given in terms of power, so that the results of [4] can be used directly. The coupling power is therefore

$$P_{ab} = \tfrac{1}{2}a^+a^{+*} - \tfrac{1}{2}a^-a^{-*} = \tfrac{1}{2}b^+b^{+*} - \tfrac{1}{2}b^-b^{-*}, \tag{2}$$

where the asterisk denotes the complex conjugate. Using the transmission relations of equation (1) gives

$$P_{ab} = \tfrac{1}{2}T^2a^+a^{+*} - \tfrac{1}{2}T^2b^-b^{-*} - \tfrac{1}{2}(tr_a^*a^{+*}b^- + t^*r_aa^+b^{-*}). \tag{3}$$

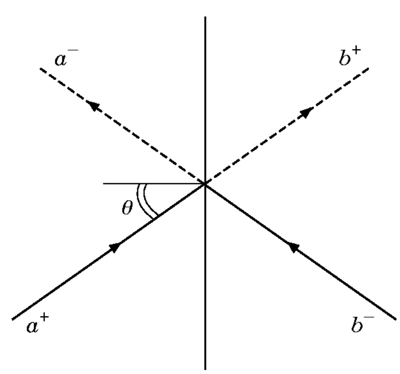

Figure 1. Wave transmission and reflection at a boundary between two subsystems.

This can be written as

$$P_{ab} = P_{sea} - P_{coh}, \tag{4}$$

where

$$P_{sea} = T^2 P_{inc,a} - T^2 P_{inc,b}, \qquad P_{inc,a} = \tfrac{1}{2} a^+ a^{+*}, \qquad P_{inc,b} = \tfrac{1}{2} b^- b^{-*} \tag{5}$$

is the coupling power normally assumed in SEA, $P_{inc,a}$ and $P_{inc,b}$ being the powers associated with the two incident waves individually, and where $T = |t|$. The coherent power is

$$P_{coh} = \tfrac{1}{2}(t r_a^* a^{+*} b^- + t^* r_a a^+ b^{-*}) \tag{6}$$

and hence depends on the relative phase of the two incident waves, and may be positive or negative, depending on this phase.

For discrete frequency excitation of a specific system by a single source, the wave amplitudes are of course coherent. It is normally assumed, however, that when frequency average (or ensemble average) powers are taken, the net coherent power is negligibly small. The arguments put forward for neglecting the net coherent power arise from the observation that the relative phase of a^+ and b^- varies rapidly with frequency and averages to zero, and P_{coh} is then assumed to tend to zero when frequency averaged [1, 5]. However, this argument fails to acknowledge that the magnitudes and phases of a^+ and b^- are not independent—certain phases correspond broadly to system resonance, and hence large wave amplitudes, and therefore give a disproportionate contribution to the frequency average of the coherent power, which can be substantial. This is explored below.

This paper concerns ensemble average powers and frequency average powers taken over wide bandwidths. Individual systems will respond differently to these averages for a number of reasons (e.g., a finite number of modes in the frequency band or coherence effects due to point force excitation) but these differences are not investigated—the paper is concerned with some reasons why these averages differ from those suggested by traditional SEA models.

The ensemble is defined as follows. As a wave circumnavigates a subsystem, so it experiences a phase change θ. The response of the system depends on $\theta \bmod 2\pi$. It is assumed here that the subsystems are drawn from an ensemble such that $\theta \bmod 2\pi$ for each subsystem is independent, random and uniformly distributed in the range $\{-\pi, \pi\}$. All other properties are assumed constant across the ensemble, consequent ensemble averages being found by integrating individual system responses over all possible θ. For the two-subsystem case, for example, the ensemble average coupling power is given by

$$\langle P_{ab} \rangle = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_{ab}(\theta_a, \theta_b)\, \mathrm{d}\theta_a\, \mathrm{d}\theta_b \tag{7}$$

where $P_{ab}(\theta_a, \theta_b)$ is the coupling power for the ensemble member system which has subsystem phases θ_a and θ_b. When frequency averages are considered, one in effect integrates $P_{ab}(\theta_a, \theta_b)$ over a range of frequency. Generally, the parameters upon which the coupling power depends vary slowly with frequency, and will here be assumed to be frequency independent, except for the subsystem phases, which vary rapidly with frequency in a manner that depends on the subsystem modal densities. Over a wide enough bandwidth the subsystem phases vary in such a way that all possible combinations (θ_a, θ_b) occur with equal probability. Certain special cases are thus excluded, one example being that where the subsystems are identical, so that $\theta_a = \theta_b$ always. This case was considered in [5, 6], while [6] also describes other cases. In modal terms, this is equivalent to assuming that uncoupled natural frequencies occur randomly in the subsystems, so that in any particular realization, over a wide enough frequency band there is a uniform, random mixture of uncoupled natural frequencies. If one excludes these special cases, then over a wide enough bandwidth frequency averages and ensemble averages are equal.

In [7] the SEA of a system comprising two one-dimensional subsystems was examined using a wave approach and some of the results are reported here. The subsystems are coupled at one end, their other ends being conservatively supported. The strength of coupling was found to be determined by the parameter γ, where $\gamma^2 \approx T^2/\mu_a\mu_b$, where $\mu = 2\varepsilon kl$ indicates the effects of damping, $k(1 - \mathrm{i}\varepsilon)$ and l being the subsystem wavenumber (complex, to include damping) and length (a further parameter δ is, in rare cases, also involved). This parameter indicates the relative importance of transmission and damping effects on the amplitude of waves traversing the system and distinguishes between two regimes in which the physical behaviour of the coupled system is qualitatively different. When $\gamma \gg 1$ transmission effects dominate, the coupling is strong and energy is shared between the two subsystems. When $\gamma \ll 1$ the coupling is weak, damping effects dominate and energy in effect leaks from the excited subsystem through the coupling. In [8] another qualitative change was seen to occur at a critical value of the coupling transmission coefficient which, for reverberant systems, is equal to γ. If T is small enough such that $\gamma < 1$, then in [8] it was seen that (broadly) the coupling energy flows are a maximum at those frequencies which correspond to the natural frequencies of the uncoupled subsystems. For $\gamma > 1$, on the other hand, peak energy flows occur at the coupled natural frequencies of the system. From an alternative perspective, consider the case of two subsystems with similar natural frequencies. The coupling introduces a shift in the natural frequencies of the system: if the coupling is weak, this shift is smaller than the bandwidth of the modes of the system, so that peak energy flows can still be associated with natural frequencies in the uncoupled state; if the coupling is strong, the shift is greater than the bandwidth, and coupled natural frequencies become important.

There are close links between γ and other, proposed, criteria for weak coupling. One physically appealing definition [9] is that when weakly coupled the dissipated powers should be much greater than the coupling powers, i.e., that $\eta_{ab}/\eta_a \ll 1$, where η_a and η_{ab} are the damping and coupling loss factors. Although closely related, they are not equivalent. If the coupling is weak in the sense $\gamma < 1$, then $\eta_{ab}/\eta_a < 1$ also, but the reverse is not true: however, the differences are generally not great. A more general definition [10, 11] is that the Green function of the driven subsystem in the coupled state should be approximately equal to that when uncoupled. γ is consistent with Langley's definition if by "approximately equal" it is meant that the shifts in natural frequencies between the uncoupled and coupled states are smaller than the bandwidth.

In the next section the coupling power and coherence effects are considered for dynamically one-dimensional subsystems. Analytical expressions for P_{coh} are found for systems comprising only two subsystems, and some conclusions are drawn for systems with more subsystems. Coherence effects are generally large when the coupling is strong. Then two-dimensional systems are discussed and comments made regarding the effects of geometric irregularity.

2. DYNAMICALLY ONE-DIMENSIONAL SUBSYSTEMS

A dynamically one-dimensional subsystem is one in which there is only one energy-propagating wave mode. Examples include rods in torsion and beams in bending, if near fields are neglected, but the subsystems need not be physically one-dimensional. A system comprising two such subsystems is shown in Figure 2. It is assumed that excitations applied to the two subsystems are statistically independent, so that it can be assumed that

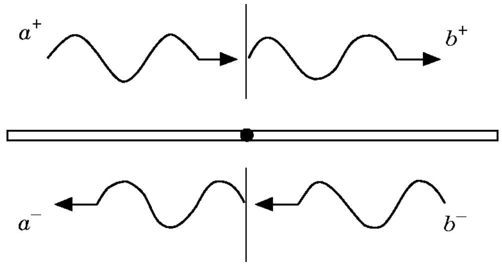

Figure 2. A system comprising two one-dimensional subsystems.

only subsystem a is excited (if both are excited, the excitations can be considered one at a time and their effects superimposed)—these excitations cause truly incoherent waves.

Waves travelling away from the coupling will be reflected from distant parts of the system (e.g., another subsystem or the far boundary of the subsystem) and contribute to the waves travelling towards the coupling. Thus,

$$a^+ = e_a + \rho_a a^-, \qquad b^- = \rho_b b^+, \qquad \rho_a = \mathrm{e}^{-\mu_a}\,\mathrm{e}^{-i\theta_a}, \qquad \rho_b = \mathrm{e}^{-\mu_b}\,\mathrm{e}^{i\theta_b}, \tag{8}$$

where ρ is a generalized reflection coefficient, μ incorporates the effects of dissipation, and θ is the phase change experienced by the wave and includes a component due to phase changes experienced at the coupling. In equation (8), e_a is the amplitude of the excited wave which appears at the coupling. This is the direct field at the coupling. From [12] it follows that

$$a^+ = \frac{1-\rho_b r_b}{\Delta}\,e_a, \qquad b^- = \frac{\rho_b t}{\Delta}\,e_a, \qquad \Delta = 1 - R\rho_a - R\rho_b + \rho_a\rho_b, \tag{9}$$

where $R = |r_a| = |r_b|$.

The coherent power follows from equations (6) and (9) and is given by

$$P_{coh} = 2T^2 R\,\mathrm{e}^{-\mu_b}\,\frac{(R\,\mathrm{e}^{-\mu_b} - \cos\theta_b)}{\Delta^*\Delta}\,P_{dir}, \tag{10}$$

where $P_{dir} = e_a^* e_a/2$ is the direct power at the coupling, while the powers incident upon the coupling are

$$P_{inc,a} = \frac{(1 + R^2\,\mathrm{e}^{-2\mu_b} - 2R\,\mathrm{e}^{\mu_b}\cos\theta_b)}{\Delta^*\Delta}\,P_{dir}, \qquad P_{inc,b} = \frac{T^2\,\mathrm{e}^{-2\mu_b}}{\Delta^*\Delta}\,P_{dir}. \tag{11}$$

As the frequency increases, so the phases θ_a and θ_b change, typically rapidly, since the subsystems are normally fairly long compared to a wavelength. However, if the damping is light there are also very large variations in the wave magnitudes, which tend to be particularly large if $\theta_{a,b} \bmod 2\pi \approx 0$. There are consequent variations in the coherent power.

It is now assumed that P_{dir} is a constant, independent of the subsystem phases. For ensemble averages this implies that the excitation level does not vary across the ensemble, while for frequency averages it implies that the excitation level is frequency independent and that there are no variations which depend on the particular spatial distribution of the excitation—"rain-on-the-roof" satisfies such assumptions.

The ensemble average and the broadband frequency average powers, in the sense defined above (i.e., the averages over θ_a and θ_b) are

$$\langle P_{ab}\rangle = \frac{T^2}{1-\mathrm{e}^{-2\mu_a}}\frac{1}{\sqrt{1+\gamma^2}\sqrt{1+\delta^2}}P_{dir},$$
$$\langle P_{inc,b}\rangle = \frac{T^2\,\mathrm{e}^{-2\mu_b}}{(1-\mathrm{e}^{-2\mu_a})(1-\mathrm{e}^{-2\mu_b})}\frac{1}{\sqrt{1+\gamma^2}\sqrt{1+\delta^2}}P_{dir},$$
$$\langle P_{coh}\rangle = \frac{T^2}{1-\mathrm{e}^{-2\mu_a}}\left[1-\frac{(1+2T^2(\mathrm{e}^{2\mu_a}-1)^{-1}-\gamma\,\delta)}{\sqrt{1+\gamma^2}\sqrt{1+\delta^2}}\right]P_{dir}, \tag{12}$$

where $\langle P_{ab}\rangle$ is given in [8], where $\langle P_{inc,a}\rangle$ follows from equations (4) and (5) and where

$$\gamma = T\frac{\cosh\mu_d}{\sqrt{\sinh\mu_a\sinh\mu_b}} \approx \frac{T}{\sqrt{\mu_a\mu_b}};$$
$$\delta = T\frac{\sinh\mu_d}{\sqrt{\sinh\mu_a\sinh\mu_b}} \approx \frac{T\mu_d}{\sqrt{\mu_a\mu_b}};$$
$$\mu_d = \frac{\mu_a-\mu_b}{2} \tag{13}$$

are the coupling strength parameters γ and δ, together with their approximations for reverberant subsystems for which μ_a and μ_b are small. (Since large γ and/or strong coupling necessarily imply reverberant subsystems, these simplified expressions will be used throughout the discussion.)

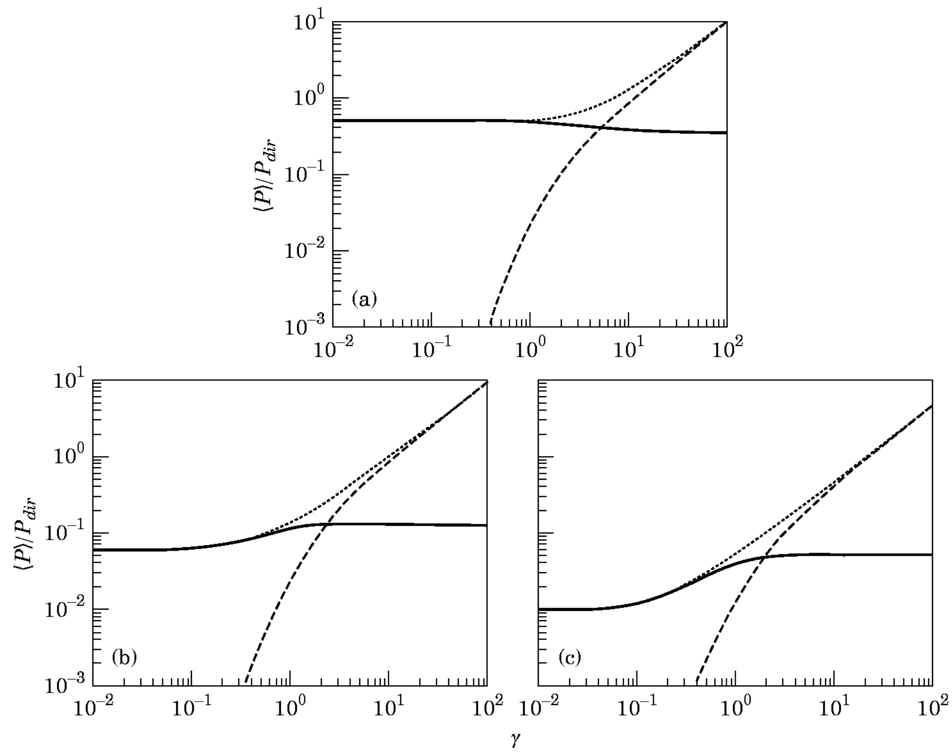

Figure 3. Ensemble average powers as a function of γ: $\mu_a = \mu_b$. (a) $T=\sqrt{0{\cdot}5}$; (b) $T = 0{\cdot}25$; (c) $T = 0{\cdot}1$. ——, $\langle P_{ab}\rangle$; ----, $\langle P_{coh}\rangle$; . . . . . . , $\langle P_{sea}\rangle$.

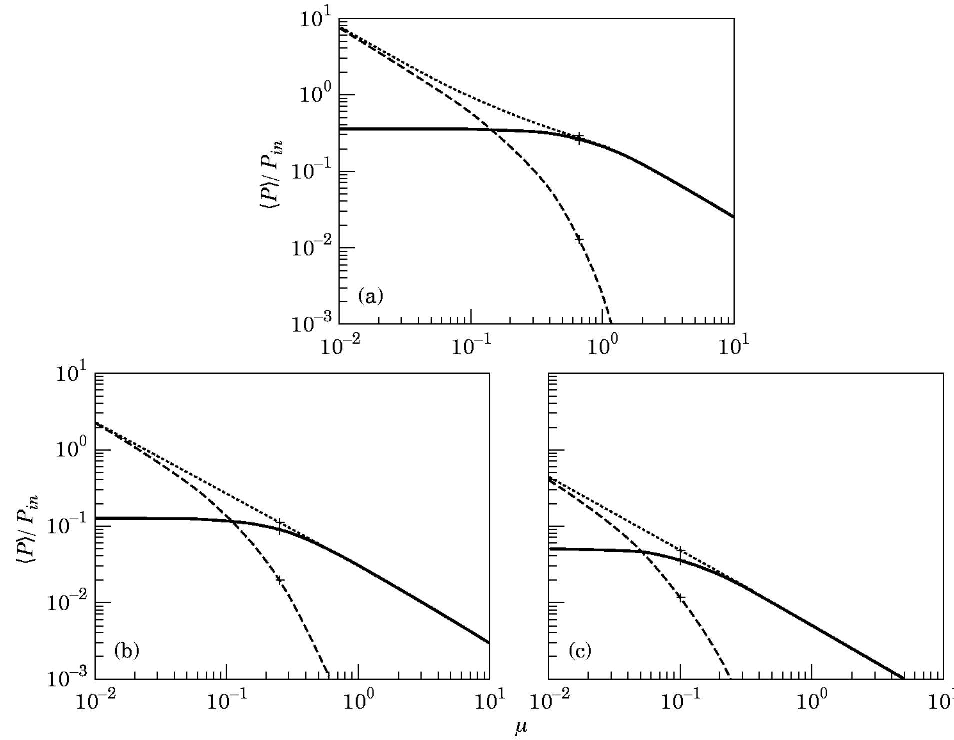

Figure 4. Ensemble average powers as a function of μ: $\mu_a = \mu_b = \mu$. (a) $T = \sqrt{0{\cdot}5}$; (b) $T = 0{\cdot}25$; (c) $T = 0{\cdot}1$. ——, $\langle P_{ab} \rangle$; ----, $\langle P_{coh} \rangle$; . . . . . . , $\langle P_{sea} \rangle$. $\gamma = 1$ at points marked $+$.

2.1. SYSTEMS COMPRISING TWO SUBSYSTEMS

Now consider the case in which the system comprises just two subsystems, the far ends of which are conservatively supported. If the subsystems are of length l, then $\theta = 2kl + \theta_0$, $\mu = 2\varepsilon kl$ and θ_0 is a phase angle associated with reflection from the ends of the subsystem [12]. Also $\mu = \pi M$, where M is the modal overlap (based on the half-power bandwidth).

In Figure 3 are shown $\langle P_{coh} \rangle$, $\langle P_{sea} \rangle = T^2(\langle P_{inc,a} \rangle - \langle P_{inc,b} \rangle)$ and the net coupling power $\langle P_{ab} \rangle$ per unit direct power P_{dir} as a function of the coupling parameter γ. For the cases shown, $\mu_a = \mu_b$ and hence $\delta = 0$. The ensemble averaged coherent power is always positive, indicating that $\langle P_{sea} \rangle$ is an overestimate. A clear change is evident at the transition from weak to strong coupling. When the coupling is weak is the sense $\gamma < 1$ (i.e., when dissipation effects dominate transmission effects), $\langle P_{coh} \rangle$ decreases rapidly with decreasing γ, is small compared to $\langle P_{ab} \rangle$ and $\langle P_{sea} \rangle \approx \langle P_{ab} \rangle$. For strong coupling, on the other hand (i.e., $\gamma > 1$) the coherent power is significant and may be very much larger than the net coupling power. In this circumstance $\langle P_{sea} \rangle \gg \langle P_{ab} \rangle$. This is due to the fact that when the coupling is strong, the response is related to the global behaviour of the system: waves propagate freely through the whole system, and hence their amplitudes at different locations are strongly coherent, and the coupled modal behaviour of the whole system is important. When ensemble or frequency averaging equation (6), the wave amplitudes themselves fluctuate depending on the subsystem phases, and, for strong coupling, *both* are large at those phase values which correspond to resonance of the coupled system. Finally, the coherent power is relatively more important for smaller T; for larger T, $\gamma = 1$ is a somewhat conservative estimate of coupling strength.

The powers per unit input power as a function of μ_a are shown in Figure 4. It is assumed that subsystem a is excited by rain-on-the-roof, in which case

$$P_{dir} = \frac{1 - e^{-2\mu_a}}{2\mu_a} \langle P_{in} \rangle. \tag{14}$$

The points at which $\gamma = 1$ are marked. Increasing μ_a corresponds to increasing levels of damping and decreasing strength of coupling. (In practical structures, typically, $\mu \sim 0{\cdot}1$ for one-dimensional subsystems.) A clear change in P_{ab} is evident at the transition from strong coupling (energy sharing) to weak coupling (energy leakage).

If the far boundaries of the subsystems are not conservatively supported, then one can write $\mu_{tot} = \mu + \mu'$, where $\mu = 2\varepsilon kl$ again and where the magnitude of the reflection coefficient of the boundary is $\exp(-\mu')$. If μ' is constant across the ensemble then the above relations hold in terms of μ_{tot}, with the exception that the direct power per unit input power is reduced by the factor $(1 + \exp(-\mu_{a,tot}))/(1 + \exp(-\mu_a))$ due to the absorption at the boundary of subsystem a. The coupling strength between the subsystems can then be estimated from $\gamma^2 = T^2/\mu_{a,tot}\mu_{b,tot}$ and is reduced by the energy dissipation at the boundaries.

Thus γ is a measure of coupling strength for this system: it separates qualitatively distinct regimes of energy leakage (weak coupling) and energy sharing; in modal terms, it separates between regimes in which peak coupling powers are associated with uncoupled modal behaviour (weak coupling) and coupled, global modal behaviour; in wave terms it distinguishes between regimes in which dissipation effects are dominant (weak coupling) and in which transmission effects are dominant, and regimes in which the coherent power is insignificant compared to the net coupling power (weak coupling) and in which it is substantial.

When the coupling is weak, therefore, the average coherent power for this system is negligible, and the ensemble average net coupling power is given by the normal SEA expression, $\langle P_{sea} \rangle$. For an individual system this is not normally the case of course. As an example, in Figure 5 are shown P_{ab}, P_{sea} and P_{coh} as a function of subsystem phases for a case of weak coupling. P_{ab} is large whenever $\theta_a = 0$ or $\theta_b = 0$, corresponding to uncoupled

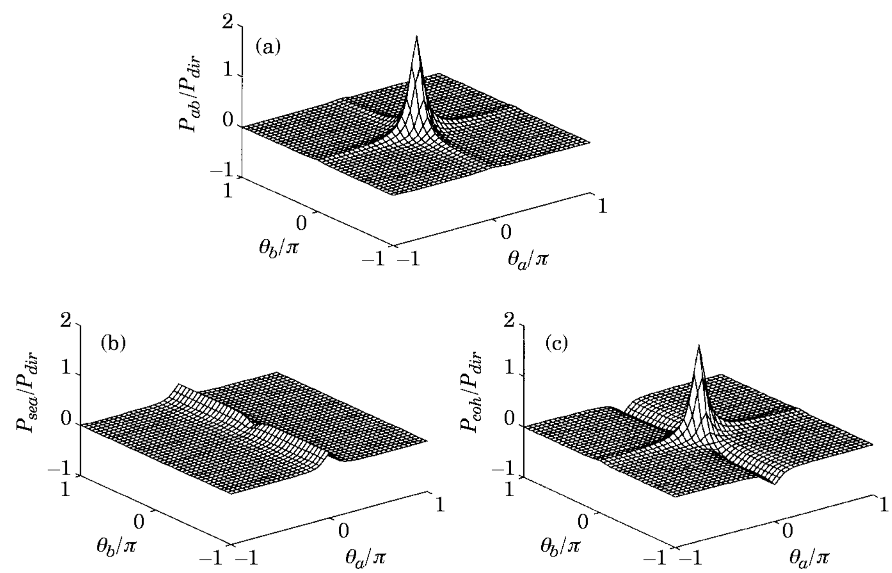

Figure 5. Powers as a function of the subsystem phase θ_a and θ_b, $\mu_a = \mu_b = 0{\cdot}2$, $T = 0{\cdot}1$, $\gamma = 0{\cdot}25$. (a) P_{ab}; (b) P_{sea}; (c) $-P_{coh}$ (note the sign).

resonance, and particularly large when both phases are zero. P_{sea} has different characteristics, however, being large typically when $\theta_a = 0$, at which phase the excited subsystem is resonant and hence the input power and $P_{inc,a}$ tend to be large. P_{coh} shows both these characteristics, is often larger than P_{sea} and has a large peak when $\theta_a = \theta_b = 0$. When the frequency average response of an individual system is found over a finite frequency band, therefore, coherence effects will be one source of difference between this average and the ensemble or broadband average, and hence a cause of variability in SEA estimates.

2.2. OTHER CASES

It is possible to draw conclusions about the behaviour of systems comprising more than two dynamically one-dimensional subsystems. These additional subsystems will affect the energy flow through the coupling. Consider, for example, the case in which a third subsystem c is connected to the far boundary of subsystem b. Then, when the effects of wave reflection at the coupling between subsystems b and c are included, ρ_b becomes

$$\rho_b' = \mathrm{e}^{-\mu_b'}\,\mathrm{e}^{-\mathrm{i}\theta_b'} = \mathrm{e}^{-\mu_b}\,\mathrm{e}^{-\mathrm{i}\theta_b} R_c \frac{(1-\rho_c/R_c)}{(1-R_c\rho_c)} \tag{15}$$

where R_c is the magnitude of the reflection coefficient of the coupling between subsystems b and c, where

$$\rho_c = \mathrm{e}^{-\mu_c}\,\mathrm{e}^{-\mathrm{i}\theta_c} \tag{16}$$

and where phase changes associated with reflection at the additional coupling are incorporated into the subsystem phases θ_b and θ_c.

Clearly, the additional subsystem acts as a dissipative boundary, as far as subsystem b is concerned. However, the situation is complicated because, since ρ_c is complex, the magnitude and phase of ρ_b' depend on θ_c. Thus while θ_b and θ_c are random and uniformly probable in the ensemble, θ_b' is not. If subsystem c is lightly damped, θ_b' is approximately equal to θ_b except at those phases θ_c which correspond broadly to uncoupled resonance of subsystem c: at such phases subsystem c draws off energy, hence increasing μ_b'. While the analysis may be complicated [13], the result as far as ensemble averaging is concerned is that, after integrating over the ensemble phase θ_c, the net result of the third subsystem is to introduce some element of dissipation at the boundary of subsystem b. The effects of coherence and the strength of coupling between subsystems a and b are not then greater than those were subsystem c replaced by a conservative boundary, and hence the parameter γ can be used to give a conservative (upper bound) estimate of the strength of coupling.

The situation becomes even more complex where subsystems a and b form part of a larger network of dynamically one-dimensional subsystems, since the dynamics of every element in the network are potentially significant. However, it can be concluded by a similar argument that a conservative estimate of the strength of coupling (and whether or not coherence effects are important) for two directly coupled subsystems in such a network, one of which is excited, can be obtained by replacing all additional subsystems with conservative couplings.

The analysis above does not hold if neither of subsystems a and b is excited, although the situation will be similar qualitatively. Suppose that a third subsystem, attached at the far end of subsystem a, is excited. Then P_{dir} will depend significantly on the subsystem properties (in a similar manner to P_{ab} when subsystem a is excited) and will hence be far from uniform across the ensemble.

Finally, the conclusions above still hold if cycles exist. Here, energy may flow from subsystem a to subsystem b and later back to subsystem a through an additional coupling point (for example, if the ends of one subsystem were to be coupled to the ends of the other).

3. TWO-DIMENSIONAL SYSTEMS

Consider two, two-dimensional subsystems jointed along an edge as shown in Figure 1 (the discussion applies equally to three-dimensional, surface coupled subsystems). The wave fields in each will generally comprise many components, each of which propagates in a direction at an angle θ to the normal to the coupling and has a particular trace wavenumber $k_t = k \sin\theta$. Equations (4) and (5) can be written for each direction of propagation, so that

$$P_{ab}(\theta) = T^2(\theta)P_{inc,a}(\theta) - T^2(\theta)P_{inc,b}(\theta) - P_{coh}(\theta) \tag{17}$$

where it is recognized that the coupling transmission coefficient depends on the incident angle θ. In the normal SEA approach the coherent power is assumed to be negligible and the wave fields assumed to be diffuse in the sense that the incident powers do not depend on θ. The total coupling power is found by integrating (17) over all possible incident angles. The wave intensity method [14, 15] relaxes the assumption of a diffuse incident field—this can make significant differences to SEA predictions, since the coupling will often preferentially transmit certain components.

3.1. REGULAR SYSTEMS

A regular subsystem is defined here to be one for which the wave components are independent. A component of given trace wavenumber which enters the subsystem from the coupling is reflected and arrives back at the coupling with the same trace wavenumber k_t (or $\pi - k_t$), whereas for an irregular subsystem it is scattered into components with other trace wavenumbers. Regular subsystems are typically physically uniform and geometrically rectangular.

A regular system is one which comprises uniformly coupled, regular subsystems. An example is that of two, rectangular, edge-coupled, simply supported plates [16]. Regular systems can therefore be regarded as comprising a set of independent, dynamically one-dimensional component systems in parallel. To each of these one-dimensional systems (the nth, say) can be ascribed a trace wavenumber and values for T_n, $\mu_{a,n}$ and $\mu_{b,n}$ which describe the transmission and reflection behaviour within the component system. Hence, from equation (13), a coupling parameter γ_n can be defined which indicates the strength of coupling of the nth component, as discussed in the previous section.

The net coupling, incident and coherent powers will be a superposition of those of each component. These will typically be either only weakly coupled, or a mixture of strongly ($\gamma_n > 1$) and weakly ($\gamma_n < 1$) coupled components, and in the latter case the strongly coupled components tend to dominate the net coupling and coherent powers. The net coherent power will be negligible if the coherent power for each component is negligible; that is, if each component is weakly coupled. Thus a conservative condition for the coherent power to be negligible, and the system as a whole to be weakly coupled, is $\max(\gamma_n) < 1$.

3.2. IRREGULAR SYSTEMS

In irregular systems there is a "mixing" of wave components. Irregularity can arise either from geometric irregularity in the subsystems, so that the subsystem scatters a wave component with one trace wavenumber into others, or from the coupling being non-uniform, so that waves incident upon the coupling are scattered into components with different trace wavenumbers.

In either case there is a tendency for the coherent power to be reduced, so that the traditional wave estimate of coupling power becomes more accurate. (There is also a tendency for the fields to become more diffuse.) For example, suppose in Figure 1 that a wave a^+, incident with a trace wavenumber k_a, is scattered partly into a coherent reflection b^- with a different trace wavenumber k_b. The product a^+b^{-*} in equation (6) then varies with distance y along the coupling as $\exp(i\Delta ky)$, where $\Delta k = k_b - k_a$. The effects of coherence between these specific components thus vary along the coupled edge and the total coherent power, integrating along the length d of coupling, decreases as $\sin(\Delta kd)/\Delta kd$.

For the case of two subsystems coupled along a line, the wave fields can be decomposed into a Fourier series along the line of coupling, in a manner analogous to [16]. Equations (8) can be writen as

$$\mathbf{a}^+ = \rho_a\mathbf{a}^- + \mathbf{e}_a, \qquad \mathbf{b}^+ = \rho_a\mathbf{b}^-, \qquad \mathbf{a}^- = \mathbf{r}_a\mathbf{a}^+ + \mathbf{t}\mathbf{b}^-; \qquad \mathbf{b}^+ = \mathbf{t}\mathbf{a}^+ + \mathbf{r}_b\mathbf{b}^-, \qquad (18)$$

where $\mathbf{a}$ and $\mathbf{b}$ are now vectors of Fourier coefficients and irregularity introduces off-diagonal terms into ρ, $\mathbf{r}$ and $\mathbf{t}$. The full behaviour is complex and is beyond the scope of this paper. However, for uniform coupling $\mathbf{r}$, $\mathbf{t}$ diagonal) and for weakly irregular subsystems (ρ diagonally dominant, so that energy scattered from one trace wavenumber component to another, then later back to the first is negligible), the net coherent power is dominated by the diagonal elements of ρ (i.e., the initial coherent reflections). In this case a criterion for weak coupling is provided by $\max(\gamma_n) < 1$, where the μ_n are calculated from the diagonal elements of ρ. Hence irregularity tends to scatter a wave component into components with other trace wavenumbers, to reduce the magnitude of these diagonal elements, increase μ and decrease the strength of coupling.

4. CONCLUDING REMARKS

This paper was concerned with the effects of wave coherence, which account for differences between the coupling power between two subsystems and that power which is normally assumed, for instance in the traditional approach to SEA. While coherence can of course be very important in determining the discrete frequency power for an individual system, its effects do not necessarily average to zero when either ensemble or broadband frequency averages are taken, and in particular the net coherent power can be substantial if the coupling is strong. This requires that a wave travelling away from the coupling is reflected back to the coupling from other regions of the structure and that strong enough coherent reflected waves exist on both sides of the coupling.

For two, one-dimensional subsystems, the average coherent power and the coupling strength was seen to depend on the parameter $\gamma = T/\sqrt{\mu_a\mu_b}$, which relates transmission and dissipation effects. For weak coupling ($\gamma < 1$) dissipation effects dominate and coherence effects, when averaged, are small. For strong coupling ($\gamma > 1$), however, waves are transmitted freely through the system and the net effects of coherence can be very large. More general cases of coupled, one-dimensional subsystems and the use of this parameter as a measure of coupling strength were then discussed. Next, two-dimensional subsystems were considered and the effects of irregularity were seen to decrease the average effects of coherence.

Broadly, γ, or $\max(\gamma)$ if more than one wave component is present, would seem to offer a measure of coupling strength: for $\gamma < 1$ coherence effects are small and the normal SEA wave approach is likely to give accurate estimates of the ensemble or broadband coupling powers, while if $\gamma > 1$ this need not be the case. Also, γ is a somewhat conservative estimate of these net coherence effects, which is perhaps not undesirable from an engineer's perspective. Finally, it should be pointed out that SEA is robust, in that conservation of energy provides a "safety valve" in the strong coupling regime. When combined with the SEA equations (giving subsystem power-balance), errors in the prediction of subsystem reponse per unit input power are substantially smaller than those in $\langle P_{sea} \rangle$, which is estimated in terms of the difference between the (incoherent) powers incident on the coupling. Thus conservation of energy and, in this strong coupling limit equipartition of energy between the subsystems, gives an upper limit to the subsystem response.

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